

ISOPERIMETRIC FUNCTIONAL INEQUALITIES VIA THE MAXIMUM PRINCIPLE: THE EXTERIOR DIFFERENTIAL SYSTEMS APPROACH

PAATA IVANISVILI AND ALEXANDER VOLBERG

ABSTRACT. The goal of this note is to give the unified approach to the solutions of a class of isoperimetric problems by relating them to the exterior differential systems studied by R. Bryant and P. Griffiths.

1. INTRODUCTION: A FUNCTION AND ITS GRADIENT

In this note we list several classical by now isoperimetric inequalities which can be proved in a unified way. This unified approach reduces them to the so-called exterior differential systems studied by Robert Bryant and Phillip Griffiths. To the best of our knowledge, this is the first article where this connection is used.

Let $d\gamma(x)$ be the standard n -dimensional Gaussian measure $d\gamma(x) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{|x|^2}{2}} dx$. Set $\Omega \subset \mathbb{R}$ to be a closed convex set and let $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$. By symbol $C_0^\infty(\mathbb{R}^n; \Omega)$ we denote the smooth, compactly supported functions on \mathbb{R}^n with values in Ω . We prove the following theorem:

Theorem 1. If a real valued function $M(x, y)$ is such that $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ and it satisfies the differential inequalities

$$\begin{pmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} \leq 0 \quad (1.1)$$

then for any $f \in C_0^\infty(\mathbb{R}^n; \Omega)$ we have

$$\int_{\mathbb{R}^n} M(f, \|\nabla f\|) d\gamma \leq M \left(\int_{\mathbb{R}^n} f d\gamma, 0 \right). \quad (1.2)$$

One can obtain the similar result for uniformly log-concave probability measures, and the short way to see this is based on the mass transportation argument. In fact, let $d\mu = e^{-U(x)} dx$ be a probability measure such that $U(x)$ is smooth and $\text{Hess } U \geq R \cdot Id$ for some $R > 0$. By the result of Caffarelli (see [1]) there exists a Brenier map $T = \nabla \phi$ for some convex function ϕ such that T pushes forward $d\gamma$ onto $d\mu$, moreover $0 \leq \text{Hess } \phi \leq \frac{1}{\sqrt{R}} \cdot Id$. We apply (1.1) to $f(x) = g(\nabla \phi(x))$ and use the fact $M_y \leq 0$ which follows from (1.1). Since $\|\nabla f(x)\| = \|\text{Hess } \phi(x) \nabla g(\nabla \phi)\| \leq \frac{1}{\sqrt{R}} \|\nabla g(\nabla \phi)\|$ we obtain:

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Corollary 1. If $M(x, y)$ satisfies $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ and (1.1) then for any $g \in C_0^\infty(\mathbb{R}^n; \Omega)$ we have

$$\int_{\mathbb{R}^n} M\left(g, \frac{\|\nabla g\|}{\sqrt{R}}\right) d\mu \leq M\left(\int_{\mathbb{R}^n} g d\mu, 0\right), \quad (1.3)$$

where $d\mu = e^{-U(x)}dx$ is a probability measure such that $\text{Hess } U(x) \geq R \cdot Id$.

In Section 1.1 we present applications of the functional inequality (1.3). In Section 2 we prove a theorem about equivalence of some functional inequalities and partial differential inequalities. Corollary 1.3 is just a consequence of this result. We will notice that our proof of Corollary 1 for general log-concave measure will not differ from the case of Gaussian measures and it will be completely self-contained (it will not need the mass transportation argument).

In Section 3 we describe solutions of (1.1) (in the important case for us when the determinant of the matrix in (1.1) is zero) by reducing it to the exterior differential system studied by R. Bryant and P. Griffiths. In Section 4 we investigate Theorem 2 in one dimensional case, and in Section 5 we present further applications.

Acknowledgement. We are very grateful to Robert Bryant from whom we learned how to solve an important for our goals non-linear PDE (see [25]). In Section 3.1 this allows us to explain how one could find the right functions $M(x, y)$ for the applications mentioned in Section 1.1.

1.1. Applications.

1.1.1. *Log-Sobolev inequalities: entropy estimates.* Log-Sobolev inequality of Gross (see [2]) states that

$$\int_{\mathbb{R}^n} |f|^2 \ln |f|^2 d\gamma - \left(\int_{\mathbb{R}^n} |f|^2 d\gamma\right) \ln \left(\int_{\mathbb{R}^n} |f|^2 d\gamma\right) \leq 2 \int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma \quad (1.4)$$

whenever the right hand side of (1.4) is well-defined and finite for complex-valued f . This implies that if f and $\|\nabla f\|$ are in $L^2(d\gamma)$ then f is in the Orlicz space $L^2 \ln L$. A proof of Gross uses *two-point inequality* which by central limit theorem establishes hypercontractivity of the Ornstein–Uhlenbeck semigroup $\|e^{t(\Delta - x \cdot \nabla)}\|_{L^p(d\gamma) \rightarrow L^q(d\gamma)} \leq 1$ for all $t \geq 0$ such that $e^{-2t} \leq \frac{p-1}{q-1}$. Then as a corollary differentiating this estimate at point $t = 0$ for $q = 2$ one obtains (1.4). Earlier than Gross similar *two-point inequality* was proved by Aline Bonami (see [3, 4]). For more on *two-point inequalities* we refer the reader to [10]. For the simple proof of hypercontractivity of Ornstein–Uhlenbeck semigroup we refer the reader to [5, 7], and also to earlier works [8, 9]. Bakry and Emery [11] extended the inequality for log-concave measures. Namely the inequality

$$\int_{\mathbb{R}^n} f^2 \ln f^2 d\mu - \left(\int_{\mathbb{R}^n} f^2 d\mu\right) \ln \left(\int_{\mathbb{R}^n} f^2 d\mu\right) \leq \frac{2}{R} \int_{\mathbb{R}^n} \|\nabla f\|^2 d\mu \quad (1.5)$$

holds for a bounded real-valued $f \in C^1$ and a log-concave probability measure $d\mu = e^{-U(x)}dx$ such that $\text{Hess } U(x) \geq R \cdot Id$. For further remarks we refer the reader to [12].

Proof of (1.5): Take

$$M(x, y) = x \ln x - \frac{y^2}{2x}, \quad x > 0 \quad \text{and} \quad y \geq 0. \quad (1.6)$$

We have

$$\begin{pmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} = \begin{pmatrix} -\frac{y^2}{x^3} & \frac{y}{x^2} \\ \frac{y}{x^2} & -\frac{1}{x} \end{pmatrix} \leq 0. \quad (1.7)$$

By Corollary 1.3 we obtain

$$\int_{\mathbb{R}^n} \left(g \ln g - \frac{1}{2R} \frac{\|\nabla g\|^2}{g} \right) d\mu \leq \left(\int_{\mathbb{R}^n} g d\mu \right) \ln \left(\int_{\mathbb{R}^n} g d\mu \right). \quad (1.8)$$

Taking $g = f^2$ for positive f and rearranging terms in (1.8) we arrive at (1.5). \square

Remark 1. The proof we just presented has an obstacle: $\frac{1}{g}$ does not make sense for $g \in C_0^\infty(\mathbb{R}^n; \mathbb{R}_+)$ and $M(x, \sqrt{y}) \notin C^2(\mathbb{R}_+ \times \mathbb{R}_+)$. In order to avoid this obstacle one has to consider $M^\varepsilon(x, y) := M(x + \varepsilon, y)$ for some $\varepsilon > 0$. Then surely $M^\varepsilon(x, y)$ will satisfies (1.1), what is more $M^\varepsilon(x, \sqrt{y}) \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$ and we can repeat the same proof as above for $M^\varepsilon(x, y)$. Finally, we just send $\varepsilon \rightarrow 0$ assuming that $\int f^2 d\mu \neq 0$ and we obtain the desired estimate. We also notice that with the same trick we weaken the requirement $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ to $M(x, \sqrt{y}) \in C^2(\text{int}(\Omega) \times \mathbb{R}_+)$ in the applications presented below, where $\text{int}(\Omega)$ denotes interior of the set Ω .

1.1.2. Poincaré inequality and spectral gap. Classical Poincaré inequality for the Gaussian measure obtained by J. Nash [26] (see p.941) states that

$$\int_{\mathbb{R}^n} f^2 d\gamma - \left(\int_{\mathbb{R}^n} f d\gamma \right)^2 \leq \int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma. \quad (1.9)$$

The inequality also says that the spectral gap i.e. the first nontrivial eigenvalue of the self-adjoint positive operator $L = -\Delta + x \cdot \nabla$ in $L^2(\mathbb{R}^n, d\gamma)$ is bounded from below by 1. If $d\mu = e^{-U(x)} dx$ is a probability measure such that $\text{Hess } U \geq R \cdot Id$ then we have

$$\int_{\mathbb{R}^n} g^2 d\mu - \left(\int_{\mathbb{R}^n} g d\mu \right)^2 \leq \frac{1}{R} \int_{\mathbb{R}^n} \|\nabla g\|^2 d\mu. \quad (1.10)$$

It is a folklore that inequality (1.10), besides of mass transportation argument, follows from the log-Sobolev inequality (1.5): apply (1.5) to the function $f(x) = 1 + \varepsilon g(x)$ where $\int g d\mu = 0$, and send $\varepsilon \rightarrow 0$. Then the left hand side of (1.5) is $2\varepsilon^2 \int g^2 d\mu + o(\varepsilon^2)$ whereas the right hand side of (1.5) is $\frac{2\varepsilon^2}{R} \int \|\nabla g\|^2 d\mu$. This gives (1.10). In [13] Brascamp and Lieb obtained the improvement of (1.10): instead of $\frac{\|\nabla g\|^2}{R}$ one can put $\langle (\text{Hess } U)^{-1} \nabla g, \nabla g \rangle$ in the right hand side of (1.10), where we assume that $\text{Hess } U$ is just positive. For a simple proof of this improvement we refer the reader to [14] (see also [15] by using Prekopa–Leindler inequality). More subtle result of Bobkov [16] in this direction says that for any log-concave probability measure $d\mu = e^{-U(x)} dx$ one can put $K \|x - \int x d\mu\|_{L^2(d\mu)}^2 \|\nabla g\|^2$ instead of $\frac{\|\nabla g\|^2}{R}$ for some universal constant $K > 0$. This implies that nonnegative operator $L = -\Delta + \nabla U \cdot x$ has a spectral gap.

In [17] Beckner found an inequality which interpolates in a sharp way between Poincaré inequality and log-Sobolev inequality. The inequality was obtained for Gaussian measures but, again, by mass transportation argument it can be easily translated to a log-concave

probability measure. Beckner–Sobolev inequality states that for $f \in L^2(d\mu)$ and $1 \leq p \leq 2$ we have

$$\int |f|^2 d\mu - \left(\int |f|^p \right)^{2/p} \leq \frac{(2-p)}{R} \int_{\mathbb{R}^n} \|\nabla f\|^2 d\mu \quad (1.11)$$

where $d\mu = e^{-U(x)} dx$ is a probability measure such that $\text{Hess } U \geq R \cdot Id$. Case $p = 1$ gives Poincaré inequality (1.10) and case $p \rightarrow 2$ after dividing the left hand side of (1.11) by $2 - p$ gives (1.5).

Proof of (1.11): Take

$$M(x, y) = x^{\frac{2}{p}} - \frac{2-p}{p^2} x^{\frac{2}{p}-2} y^2 \quad \text{where } x, y \geq 0 \quad 1 \leq p \leq 2. \quad (1.12)$$

Notice that

$$\begin{pmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} = \begin{pmatrix} -\frac{2(2-p)(1-p)(2-3p)x^{\frac{2}{p}-4}y^2}{p^4} & -\frac{4(2-p)(1-p)x^{\frac{2}{p}-3}y}{p^3} \\ -\frac{4(2-p)(1-p)x^{\frac{2}{p}-3}y}{p^3} & -\frac{4(2-p)x^{\frac{2}{p}-2}}{p^2} \end{pmatrix} \leq 0. \quad (1.13)$$

By Corollary 1 we have

$$\int_{\mathbb{R}^n} g^{\frac{2}{p}} - \frac{2-p}{p^2} g^{\frac{2}{p}-2} \frac{\|\nabla g\|^2}{R} d\mu \leq \left(\int_{\mathbb{R}^n} g d\mu \right)^{\frac{2}{p}} \quad (1.14)$$

for positive (in fact nonnegative) functions g . Now set $g = |f|^p$, and notice that $\|\nabla |f|^p\| \leq p \|\nabla f\| |f|^{p-1}$. After rearranging terms in (1.14) we obtain (1.11). \square

1.1.3. Bobkov's inequality: Gaussian isoperimetry. In [18] Bobkov obtained the following functional version of Gaussian isoperimetry. Let $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-x^2/2} dx$, and let $\Phi'(x)$ be a derivative of Φ . Set $I(x) := \Phi'(\Phi^{-1}(x))$. Then for any locally Lipschitz function $f : \mathbb{R}^n \rightarrow [0, 1]$, we have

$$I \left(\int_{\mathbb{R}^n} f d\mu \right) \leq \int_{\mathbb{R}^n} \sqrt{I^2(f) + \frac{\|\nabla f\|^2}{R}} d\mu \quad (1.15)$$

where $d\mu = e^{-U(x)} dx$ is a log-concave probability measure such that $\text{Hess } U \geq R \cdot Id$. Bobkov's proof uses a *two-point inequality*: for all $0 \leq a, b \leq 1$ we have

$$I \left(\frac{a+b}{2} \right) \leq \frac{1}{2} \sqrt{I^2(a) + \left| \frac{a-b}{2} \right|^2} + \frac{1}{2} \sqrt{I^2(b) + \left| \frac{a-b}{2} \right|^2}. \quad (1.16)$$

Iterating (1.16) appropriately and using central limit theorem Bobkov obtained (1.15) for the Gaussian measures. By the mass transportation argument one immediately obtains (1.15) for uniformly log-concave measures. Notice that $I(0) = I(1) = 0$. Testing (1.15) for $d\mu = d\gamma$ and $f(x) = \mathbb{1}_A$ where A is a Borel subset of \mathbb{R}^n one obtains Gaussian isoperimetry: for any Borel measurable set $A \subset \mathbb{R}^n$

$$\gamma^+(A) \geq \Phi'(\Phi^{-1}(\gamma(A))) \quad \text{where } \gamma^+(A) := \liminf_{\varepsilon \rightarrow 0} \frac{\gamma(A_\varepsilon) - \gamma(A)}{\varepsilon} \quad (1.17)$$

denotes Gaussian perimeter of A , here $A_\varepsilon = \{x \in \mathbb{R}^n : \text{dist}_{\mathbb{R}^n}(A, x) < \varepsilon\}$. For further remarks on (1.15) see [19]. Gaussian isoperimetry (1.17) can be derived also from Ehrhard's inequality (see for example [5]).

Proof of (1.15). Take

$$M(x, y) = -\sqrt{I^2(x) + y^2} \quad \text{where } x \in [0, 1], \quad y \geq 0. \quad (1.18)$$

We have

$$\begin{pmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} = \begin{pmatrix} -\frac{(I'(x))^2 y^2}{(I^2(x) + y^2)^{3/2}} + \frac{I(x)I''(x) + 1}{\sqrt{I^2(x) + y^2}} & y \frac{I(x)I'(x)}{(I^2(x) + y^2)^{3/2}} \\ y \frac{I(x)I'(x)}{(I^2(x) + y^2)^{3/2}} & -\frac{I^2(x)}{(I^2(x) + y^2)^{3/2}} \end{pmatrix}. \quad (1.19)$$

Notice that $I''(x)I(x) + 1 = 0$ therefore (1.19) is negative semidefinite. So by Corollary 1 we obtain

$$\int_{\mathbb{R}^n} -\sqrt{I^2(f) + \frac{\|\nabla f\|^2}{R}} d\mu \leq -I \left(\int_{\mathbb{R}^n} f d\mu \right) \quad (1.20)$$

rearranging terms in (1.20) we obtain (1.15) for differentiable $f : \mathbb{R}^n \rightarrow [0, 1]$. Notice that (1.20) still holds if $I''(x)I(x) + 1 \leq 0$ for arbitrary smooth $I(x)$. \square

1.1.4. (B) *Theorem.* The problem was proposed by W. Banaszczyk (see for example [20]) which says that given symmetric convex body $K \subset \mathbb{R}^n$ the function $\phi(t) = \gamma(e^t K)$ is log-concave on \mathbb{R} . The problem was solved in [21]: clearly one only needs to check log-concavity at one point: $(\ln \phi(t))''|_{t=0} \leq 0$. This is the same as

$$\int_{\mathbb{R}^n} \|x\|^4 d\gamma_K - \left(\int_{\mathbb{R}^n} \|x\|^2 d\gamma_K \right)^2 \leq 2 \int_{\mathbb{R}^n} \|x\|^2 d\gamma_K \quad (1.21)$$

where

$$d\gamma_K = \frac{\mathbb{1}_K(x) e^{-\|x\|^2/2} dx}{\int_K e^{-\|y\|^2/2} dy} = e^{-\|x\|^2/2 - \psi(x)} dx$$

where a convex function ψ is a constant on K and it is $+\infty$ outside of the set K . In other words one can assume that $d\gamma_K = e^{-U(x)} dx$ is a probability measure where $U(x)$ is even and such that $\text{Hess } U \geq Id$. Setting $f(x) = \|x\|^2$ then inequality (1.21) can be rewritten as follows

$$\int_{\mathbb{R}^n} f^2 d\mu - \left(\int_{\mathbb{R}^n} f d\mu \right)^2 \leq \frac{1}{2} \int_{\mathbb{R}^n} \|\nabla f\|^2 d\mu. \quad (1.22)$$

which is better than Poincaré inequality (1.10). This is a key ingredient in (B) Theorem and it was proved by Cordero-Erausquin–Fradelizi–Maurey in [21] that (1.22) holds provided that $\int_{\mathbb{R}^n} \nabla f d\mu = 0$, and $d\mu = e^{-U(x)} dx$ is a probability measure such that $\text{Hess } U \geq Id$ (which is true for $f(x) = \|x\|^2$).

If one tries to apply Corollary 1 then the right choice of the function M must be

$$M(x, y) = x^2 - \frac{y^2}{2} \quad (1.23)$$

but unfortunately this function does not satisfy (1.1). However, we want (1.22) to hold only for the functions such that $\int \nabla f d\mu = 0$ therefore one can slightly modify the proof of Theorem 1 in order to obtain (1.22). In Section 4 we will show how it works and we will present a different proof of (1.22) with the extra conditions that f is even and $d\mu$ is even (which definitely is enough for the (B) Theorem).

1.1.5. *Concluding remarks.* As we shall notice in order to use Theorem 1 for the applications to functional (and thereby isoperimetric) inequalities, there is one difficulty: one has to find the right function $M(x, y)$, for example such as (1.6), (1.12), (1.18) and (1.23). If one knows what inequality should be proved then one can try to guess what function $M(x, y)$ one has to choose: in the integrand one needs to set $g = x$ and $\|\nabla g\| = y$ and then integrand in terms of x and y will be $M(x, y)$.

In general finding $M(x, y)$ will be based purely on solving PDEs. First notice that given, for example, a convex function $f : \Omega \rightarrow \mathbb{R}$ and suppose one wants to find an optimal *error term* in the Jensen's inequality

$$0 \leq \int_{\mathbb{R}^n} f(g(x)) d\gamma - f\left(\int_{\mathbb{R}^n} g d\gamma\right) \leq \int_{\mathbb{R}^n} \text{Error}(g, \|\nabla g\|) d\gamma \quad \text{for all } g \in C_0^\infty(\mathbb{R}^n; \Omega).$$

If we find $M(x, y)$ such that $M(x, 0) = f(x)$ and $M(x, y)$ satisfies (1.1) then by Theorem 1 we can find a possible error term as follows

$$\int_{\mathbb{R}^n} f(g(x)) d\gamma - f\left(\int_{\mathbb{R}^n} g d\gamma\right) \leq \int_{\mathbb{R}^n} [M(g, 0) - M(g, \|\nabla g\|)] d\gamma. \quad (1.24)$$

In fact we would like to minimize the error term which corresponds to maximize $M(x, y)$ under the constraints (1.1) and $M(x, 0) = f(x)$. This suggests that partial differential inequality (1.1) should degenerate. Therefore we will seek $M(x, y)$ among those functions which in addition with (1.1) also satisfy *Monge–Ampère equation with a drift*:

$$\det \begin{pmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} = M_{xx}M_{yy} - M_{xy}^2 + \frac{M_y M_{yy}}{y} = 0 \quad (1.25)$$

for $(x, y) \in \Omega \times \mathbb{R}_+$.

For example in log-Sobolev (1.5) and in Bobkov's inequality (1.15) determinant of the matrices (1.7) and (1.19) are zero. In Beckner–Sobolev inequality (1.11) determinant of (1.13) is zero if and only if $p = 1, 2$. Notice that these are exactly cases when Beckner–Sobolev inequality interpolates Poincaré and log-Sobolev inequality.

In Section 3 we will show that thanks to the exterior differential systems studied by R. Bryant and P. Griffiths (see [22, 23, 24]) nonlinear equation (1.25) can be reduced (after suitable change of variables) to linear backwards heat equation. In Section 3.1 we will illustrate this on the examples

$$M(x, 0) = x \ln x, \quad M(x, 0) = x^2 \quad \text{and} \quad M(x, 0) = -I(x).$$

which correspond to log-Sobolev, Poincaré and Bobkov's inequalities.

2. FUNCTION OF SEVERAL VARIABLES $D^{\alpha} \mathbf{f}$

Let $d\mu = e^{-U(x)} dx$ be a log-concave measure such that U is smooth and $\text{Hess } U \geq R \cdot Id$. Set $L = \Delta - \nabla U \cdot \nabla$. Then $-L$ is a self-adjoint positive operator in $L^2(\mathbb{R}^n, d\mu)$, moreover by (1.10) it has a spectral gap. Let $P_t := e^{tL}$ be the corresponding semigroup generated by L . Let $\alpha = (\alpha_0, \dots, \alpha_m)$ where $\alpha_j = (\alpha_j^1, \dots, \alpha_j^n)$ is a multi index of size n and $\alpha_j^i \in \mathbb{N} \cup \{0\}$ for each $j = 0, \dots, m$ and $i = 1, \dots, n$. Let $|\alpha_j|$ be the length of the multi index i.e. $|\alpha_j| = \alpha_j^1 + \dots + \alpha_j^n$. By D^{α_j} we denote the differential operator

$$D^{\alpha_j} = \frac{\partial^{|\alpha_j|}}{\partial x_1^{\alpha_j^1} \dots \partial x_n^{\alpha_j^n}}.$$

Further we fix some *multi-multi* index $\alpha = (\alpha_0, \dots, \alpha_m)$ where each α_j is a multi index of size n as above.

Test functions $C_0^\infty(\mathbb{R}^n; \Lambda)$. Let Λ be a closed convex subset of \mathbb{R}^m . By $C_0^\infty(\mathbb{R}^n; \Lambda)$ we denote the set of test functions $\mathbf{f} = (f_0, \dots, f_m) : \mathbb{R}^n \rightarrow \Lambda$ i.e., smooth and compactly supported vector functions with values in Λ . Let

$$D^\alpha \mathbf{f} = (D^{\alpha_0} f_0, \dots, D^{\alpha_m} f_m) \quad \text{and} \quad P_t \mathbf{f} := (P_t f_0, \dots, P_t f_m).$$

Notice that we have $P_t \mathbf{f} \in \Lambda$. Indeed, take any closed half space that contains Λ i.e., $\langle x, p \rangle \leq q$ for all $x \in \Lambda$ and for some $p \in \mathbb{R}^m$ and $q \in \mathbb{R}$. Then $\langle \mathbf{f}, p \rangle \leq q$. Since $P_t q = q$, P_t is linear and positive we have $\langle P_t \mathbf{f}, p \rangle \leq q$. This implies that $P_t \mathbf{f} \in \Lambda$. Further we assume that whenever α_j is non zero multi index then projection of Ω onto j th coordinate is the whole real line \mathbb{R} . This implies that $P_t D^\alpha \mathbf{f}$ takes values in Λ .

Let $B(u_1, \dots, u_m) : \Lambda \rightarrow \mathbb{R}$ be a smooth (at least C^2) function, such that $P_t B(D^\alpha \mathbf{f})$ is well defined for all $t \geq 0$. Set

$$[L, D^\alpha] \mathbf{f} \stackrel{\text{def}}{=} ([L, D^{\alpha_0}] f_0, \dots, [L, D^{\alpha_m}] f_m) \quad \text{and} \quad \Gamma(D^\alpha \mathbf{f}) \stackrel{\text{def}}{=} \{\langle \nabla D^{\alpha_i} f_i, \nabla D^{\alpha_j} f_j \rangle\}_{i,j=0}^m$$

where $\Gamma(D^\alpha \mathbf{f})$ denotes $(m+1) \times (m+1)$, and $[A, B] = AB - BA$ denotes commutator of A and B .

Theorem 2. The following conditions are equivalent:

- (i) $\nabla B(D^\alpha \mathbf{f}) \cdot [L, D^\alpha] \mathbf{f} + \text{Tr}[\text{Hess } B(D^\alpha \mathbf{f}) \Gamma(D^\alpha \mathbf{f})] \leq 0$ for all $f \in C_0^\infty(\mathbb{R}^n; \Lambda)$.
- (ii) $P_t[B(D^\alpha \mathbf{f})](x) \leq B(D^\alpha[P_t \mathbf{f}])(x)$ for all $t \geq 0$, $x \in \mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n; \Lambda)$.

Proof. (i) implies (ii): let $V(x, t) = P_t[B(D^\alpha \mathbf{f})](x) - B(D^\alpha[P_t \mathbf{f}])(x)$. Notice that

$$\begin{aligned} (\partial_t - L)V(x, t) &= (L - \partial_t)B(D^\alpha[P_t \mathbf{f}])(x) = \\ &= \sum_j \frac{\partial B}{\partial u_j} L D^{\alpha_j} P_t f_j + \sum_{i,j} \frac{\partial^2 B}{\partial u_i \partial u_j} \nabla D^{\alpha_i} P_t f_i \cdot \nabla D^{\alpha_j} P_t f_j - \sum_j \frac{\partial B}{\partial u_j} D^{\alpha_j} L P_t f_j = \\ &= \nabla B(D^\alpha P_t \mathbf{f}) \cdot [L, D^\alpha] P_t \mathbf{f} + \text{Tr}(\text{Hess } B(D^\alpha P_t \mathbf{f}) \Gamma(D^\alpha P_t \mathbf{f})) \leq 0 \end{aligned} \quad (2.1)$$

The last inequality follows from (i) and the fact that $P_t \mathbf{f}(x) \in \Lambda$. Indeed, we can find a function $\mathbf{g} \in C_0^\infty(\mathbb{R}^n; \Lambda)$ such that $\mathbf{g} = P_t \mathbf{f}$ in a neighborhood of x and we can apply (i) to \mathbf{g} .

By maximum principle we obtain $V(x, t) \leq \sup_x V(x, 0) = 0$. Another way (without maximum principle) is that

$$V(x, t) = \int_0^t \frac{\partial}{\partial s} P_s B(D^\alpha P_{t-s} \mathbf{f}) ds = \int_0^t P_s \left[\left(L - \frac{\partial}{\partial t} \right) B(D^\alpha P_{t-s} \mathbf{f}) \right] ds, \quad (2.2)$$

and the integrand in (2.2) is non positive by (2.1).

(ii) implies (i): for all $\mathbf{f} \in C_0^\infty(\mathbb{R}^n; \Lambda)$ we have

$$\begin{aligned} 0 &\geq \lim_{t \rightarrow 0} \frac{V(x, t)}{t} = \lim_{t \rightarrow 0} \frac{V(x, t) - V(x, 0)}{t} = \frac{\partial}{\partial t} V(x, t)|_{t=0} = \\ &= \nabla B(D^\alpha \mathbf{f}) \cdot [L, D^\alpha] \mathbf{f} + \text{Tr}(\text{Hess } B(D^\alpha \mathbf{f}) \Gamma(D^\alpha \mathbf{f})). \end{aligned}$$

□

Remark 2. We notice that if one considers diffusion semigroups generated by

$$L = \sum_{ij} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j(x) \frac{\partial}{\partial x_j}$$

where $A = \{a_{ij}\}_{i,j=1}^n$ is positive then absolutely nothing changes in Theorem 2 except the matrix $\Gamma(D^\alpha \mathbf{f})$ takes the form

$$\Gamma(D^\alpha \mathbf{f}) = \{\nabla D^{\alpha_i} f_i A (\nabla D^{\alpha_j} f_j)^T\}_{i,j=0}^m.$$

2.1. Proof of Theorem 1. Consider a special case when $n = m$, $\mathbf{f} = \underbrace{(f, \dots, f)}_{n+1}$, $\alpha_0 = \underbrace{(0, \dots, 0)}_n$, $\alpha_1 = (1, 0, \dots, 0), \dots$, and $\alpha_n = (0, \dots, 0, 1)$. Then $D^\alpha \mathbf{f} = (f, \nabla f)$, and given that $L = \Delta - \nabla U \cdot \nabla$ we obtain

$$\nabla B(D^\alpha \mathbf{f}) \cdot [L, D^\alpha] \mathbf{f} = \nabla_{1, \dots, n} B (\text{Hess } U) (\nabla f)^T.$$

Here $\nabla_{1, \dots, n} B$ is a gradient of $B(u_0, \dots, u_n)$ taken with respect to u_1, \dots, u_n variables. Assume that f takes values in the closed convex set $\Omega \subset \mathbb{R}$. Take

$$B(u_0, \dots, u_n) = M \left(u_0, \sqrt{\frac{u_1^2 + \dots + u_n^2}{R}} \right),$$

where $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ satisfies (1.1). Notice that $M_y \leq 0$. Indeed, if we multiply the first diagonal entry of (1.1) by y and send $y \rightarrow 0$ we obtain $M_y(x, 0) \leq 0$. On the other hand since the second diagonal entry of (1.1) is nonpositive we obtain $M_y(x, y) \leq 0$ for all y .

Next we notice

$$\nabla_{1, \dots, n} B(D^\alpha \mathbf{f}) = \frac{M_y}{\|\nabla f\| \sqrt{R}} \nabla f.$$

Since $M_y \leq 0$ and $\text{Hess } U \geq R \cdot Id$, we have

$$\nabla B(D^\alpha \mathbf{f}) \cdot [L, D^\alpha] \mathbf{f} = \frac{M_y}{\|\nabla f\| \sqrt{R}} \nabla f (\text{Hess } U) (\nabla f)^T \leq \sqrt{R} \|\nabla f\| M_y.$$

Therefore

$$\nabla B(D^\alpha \mathbf{f}) \cdot [L, D^\alpha] \mathbf{f} + \text{Tr}(\text{Hess } B(D^\alpha \mathbf{f})) \Gamma(D^\alpha \mathbf{f}) \leq \text{Tr}(W \Gamma(D^\alpha \mathbf{f}))$$

where

$$W = \begin{bmatrix} \partial_{00}^2 B + \frac{\sqrt{R} \cdot M_y}{\|\nabla f\|} & \partial_{01}^2 B & \dots & \partial_{0n}^2 B \\ \partial_{10}^2 B & \partial_{11}^2 B & \dots & \partial_{1n}^2 B \\ \dots & \dots & \dots & \dots \\ \partial_{n0}^2 B & \partial_{n1}^2 B & \dots & \partial_{nn}^2 B \end{bmatrix}$$

where $\partial_{ij}^2 B = \frac{\partial^2 B}{\partial u_i \partial u_j}$. We will show that $W \leq 0$, and then we will obtain $\text{Tr}(W \Gamma(D^\alpha \mathbf{f})) \leq 0$ because $\Gamma(D^\alpha \mathbf{f}) \geq 0$.

We have $\partial_{00}^2 B = M_{xx}$, $\partial_{0j}^2 B = \frac{M_{xy}}{\|\nabla f\| \sqrt{R}} f_{xj}$ for all $j \geq 1$ and

$$\partial_{ij}^2 B = \frac{M_{yy}}{\|\nabla f\|^2 R} f_{xi} f_{xj} - \frac{M_y}{\|\nabla f\|^3 \sqrt{R}} f_{xi} f_{xj} + \frac{M_y \delta_{ij}}{\|\nabla f\| \sqrt{R}} \quad \text{for } i, j \geq 1$$

where δ_{ij} is Kronecker symbol.

Notice that since $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ we have that $B \in C^2(\Omega \times \mathbb{R}^n)$. If $\nabla f = 0$ then there is nothing to prove because W becomes diagonal matrix with negative entries on the diagonal. Further assume $\|\nabla f\| \neq 0$.

Now notice that

$$W = S \left(W_1 + \frac{M_y \sqrt{R}}{\|\nabla f\|} W_2 \right) S$$

where S is a diagonal matrix with diagonal $(1, \frac{\nabla f}{\|\nabla f\| \sqrt{R}})$, and

$$W_1 = \begin{bmatrix} M_{xx} + \frac{\sqrt{R} M_y}{\|\nabla f\|} & M_{xy} & \dots & M_{xy} \\ M_{xy} & M_{yy} & \dots & M_{yy} \\ \dots & \dots & \dots & \dots \\ M_{xy} & M_{yy} & \dots & M_{yy} \end{bmatrix} \quad \text{and} \quad W_2 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{\|\nabla f\|^2}{(f_{x_1})^2} - 1 & -1 & \dots & -1 \\ 0 & -1 & \frac{\|\nabla f\|^2}{(f_{x_2})^2} - 1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & -1 & \dots & -1 & \frac{\|\nabla f\|^2}{(f_{x_n})^2} - 1 \end{bmatrix}$$

It is clear that $W_1 \leq 0$ because M satisfies (1.1) at point x and $\frac{y}{\sqrt{R}}$.

For the W_2 , first notice that if $f_{x_j} \neq 0$ for all $j \geq 1$ then W_2 is well defined and $W_2 \geq 0$. Otherwise if $f_{x_j} = 0$ for some j , then consider initial expression SW_2S and notice that $SW_2S = S\tilde{W}_2S + D$, where \tilde{W}_2 is the same as W_2 except j th column and row are replaced by zeros, and D is zero matrix except the element (j, j) is equal to $\frac{1}{R}$. We again see that $SW_2S \geq 0$. Hence $M_y SW_2S \leq 0$ as soon as (1.1) holds.

Thus we have proved that if $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$, M satisfies (1.1) then by Theorem 2 we have

$$P_t M(f, \|\nabla f\|) \leq M(P_t f, \|\nabla P_t f\|) \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n; \Omega). \quad (2.3)$$

We send $t \rightarrow \infty$ and because of the fact $\|\nabla P_t f\| \leq e^{-tR} P_t \|\nabla f\|$ (see [12]) we obtain

$$\int_{\mathbb{R}^n} M(f, \|\nabla f\|) d\mu \leq M \left(\int_{\mathbb{R}^n} f d\mu, 0 \right),$$

where $d\mu = e^{-U(x)} dx$ is a probability measure.

Remark 3. It is worth mentioning but not necessary for our purposes that (2.3) also implies (1.1) in case of Gaussian measure. This follows from the fact that the matrix $\lambda \Gamma(D^\alpha \mathbf{f})$ can be an arbitrary positive definite matrix where $\lambda > 0$ and $\mathbf{f} \in C^\infty(\mathbb{R}^n; \Lambda)$. Then condition $\text{Tr}(W \Gamma(D^\alpha \mathbf{f})) \leq 0$ implies that $W \leq 0$ and this gives us condition (1.1).

3. REDUCTION TO THE EXTERIOR DIFFERENTIAL SYSTEMS AND BACKWARDS HEAT EQUATION

As we have already mentioned in Section 1.1.5 (and it also follows from the proof of Theorem 1) in order inequality (1.2) to be *sharp* we need to assume that (1.1) degenerates i.e.,

$$\det \begin{pmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} = M_{xx} M_{yy} - M_{xy}^2 + \frac{M_y M_{yy}}{y} = 0. \quad (3.1)$$

Let us make the following observation: consider 1-graph of $M(x, y)$ i.e.,

$$(x, y, p, q) = (x, y, M_x(x, y), M_y(x, y))$$

in $xypq$ -space. This is a simply-connected surface Σ in 4-space on which $\Upsilon = dx \wedge dy$ is nonvanishing but to which the two 2-forms

$$\Upsilon_1 = dp \wedge dx + dq \wedge dy \quad \text{and} \quad \Upsilon_2 = (ydp + qdx) \wedge dq$$

pull back to be zero.

Conversely, suppose given simply connected surface Σ in $xypq$ -space (with $y > 0$) on which Υ is nonvanishing but to which Υ_1 and Υ_2 pullback to be zero. The 1-form $pdx + qdy$ pull back to Σ to be closed (since Υ_1 vanishes on Σ) and hence exact, and therefore there exists a function $m : \Sigma \rightarrow \mathbb{R}$ such that $dm = pdx + qdy$ on Σ . We then have (at least locally), $m = M(x, y)$ on Σ and, by its definition, we have $p = M_x(x, y)$ and $q = M_y(x, y)$ on the surface. Then fact that Υ_2 vanishes when pulled back to Σ implies that $M(x, y)$ satisfies the desired equation.

Thus, we have encoded the given PDE as an exterior differential system on \mathbb{R}^4 . Note, that we can make a change of variables on the open set where $q < 0$: Set $y = qr$ and let $t = \frac{1}{2}q^2$. then, using these new coordinates on this domain, we have

$$\Upsilon_1 = dp \wedge dx + dt \wedge dr \quad \text{and} \quad \Upsilon_2 = (r dp + dx) \wedge dt.$$

Now, when we take an integral surface Σ on these 2-forms on which $dp \wedge dt$ is vanishing, it can be written locally as a graph of the form

$$(p, t, x, r) = (p, t, u_p(p, t), u_t(p, t))$$

(since Σ is an integral of Υ_1), where $u(p, t)$ satisfies $u_t + u_{pp} = 0$ (since Σ is an integral of Υ_2). Thus, “generically” our PDE is equivalent to the backwards heat equation, up to a change of variables. Thus the function $M(x, y)$ can be parametrized as follows

$$\begin{aligned} x &= u_p \left(p, \frac{1}{2}q^2 \right); \quad y = qu_t \left(p, \frac{1}{2}q^2 \right); \\ M(x, y) &= pu_p \left(p, \frac{1}{2}q^2 \right) + q^2 u_t \left(p, \frac{1}{2}q^2 \right) - u \left(p, \frac{1}{2}q^2 \right). \end{aligned} \quad (3.2)$$

Note that $y \geq 0$, $q = M_y \leq 0$ then $u_t(p, \frac{1}{2}q^2) \leq 0$. Let us rewrite the conditions $M_{yy} \leq 0$ and $M_{xx} + \frac{M_y}{y} \leq 0$ in terms of $u(p, t)$. In other words we want q_y and $p_x + \frac{q}{y} \leq 0$. We have

$$0 = u_{pp}p_y + u_{pt}qq_y \quad \text{and} \quad 1 = q_y u_t + qp_y u_{tp} + q^2 q_y u_{tt}.$$

Then

$$1 = q_y u_t + q^2 q_y \frac{u_{pt}^2}{u_{pp}} + q^2 q_y u_{tt} \quad \text{and} \quad M_{yy} = q_y = \frac{u_t}{u_t^2 - 2t(u_{tt}u_{pp} - u_{pt}^2)}.$$

Thus the negative definiteness of the matrix (1.1) (if its determinant is known to be zero) is equivalent to

$$u_t^2 - 2t \det(\text{Hess } u) \geq 0. \quad (3.3)$$

Let us show that the function $u(p, t)$ must satisfy a boundary condition:

$$u(f'(x), 0) = xf'(x) - f(x) \quad \text{for} \quad x \in \Omega \quad \text{where} \quad f(x) = M(x, 0). \quad (3.4)$$

Indeed, we know that $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ therefore $M_y(x, 0) = 0$. By choosing $y = 0$ in (3.2), we have $q = 0$, and we obtain the desired boundary condition:

$$M(x, 0) = xM_x(x, 0) - u(M_x(x, 0), 0).$$

Now it is clear how to find the function $M(x, y)$ provided that $M(x, 0)$ is given: First we try

to find a function $u(p, t)$ such that

$$u_{pp} + u_t = 0, \quad u_t \leq 0, \quad (3.5)$$

$$u(M_x(x, 0), 0) = xM_x(x, 0) - M(x, 0) \quad x \in \Omega, \quad (3.6)$$

$$u_t^2 - 2t \det(\text{Hess } u) \geq 0. \quad (3.7)$$

Then a candidate for $M(x, y)$ will be given by (3.2).

3.1. Back to the applications. Further we assume that we know the expression $M(x, 0)$ and we would like to restore the function $M(x, y)$ which satisfies conditions of Theorem 1, PDE (3.1) and hence it gives us inequality (1.3), or the error term in Jensen's inequality (see Section 1.1.5 for the explanations).

3.1.1. Gross function. In this case we have $M(x, 0) = x \ln x$. Condition (3.6) can be rewritten as follows $u(p, 0) = e^{p-1}$ for all $p \in \mathbb{R}$. If we set $D = \frac{\partial^2}{\partial p^2}$ then

$$u(p, t) = e^{-tD} e^{p-1} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} e^{p-1} = e^{p-t-1} \quad \text{for all } t \geq 0.$$

Clearly $u(p, t)$ satisfies (3.7) because $\det(\text{Hess } u) = 0$. Notice that we have $u_t < 0$,

$$\begin{cases} x = e^{p-\frac{q^2}{2}-1}; \\ y = -qe^{p-\frac{q^2}{2}-1}; \end{cases} \quad \text{then} \quad \begin{cases} q = -\frac{y}{x}; \\ p = \ln x + \frac{y^2}{2x^2} + 1. \end{cases}$$

Therefore we obtain

$$M(x, y) = xp + qy - u\left(p, \frac{1}{2}q^2\right) = x \ln x + \frac{y^2}{2x} + x - \frac{y^2}{x} - x = x \ln x - \frac{y^2}{2x}.$$

3.1.2. Nash's function. In this case we have $M(x, 0) = x^2$. Condition (3.6) takes the form $u(p, 0) = \frac{p^2}{4}$ for all $p \in \mathbb{R}$. Then

$$u(p, t) = e^{-tD} \frac{p^2}{4} = (1 - tD) \frac{p^2}{4} = \frac{p^2}{4} - \frac{t}{2} \quad t \geq 0.$$

$u(p, t)$ satisfies (3.7) because $\det(\text{Hess } u) = 0$. We have $u_t < 0$

$$\begin{cases} x = \frac{p}{2}; \\ y = -\frac{q}{2}; \end{cases} \quad \text{then} \quad \begin{cases} p = 2x; \\ q = -2y. \end{cases}$$

We obtain

$$M(x, y) = 2x^2 - 2y^2 - (x^2 - y^2) = x^2 - y^2.$$

3.1.3. Bobkov's function. In this case we have $M(x, 0) = -I(x)$. Condition (3.6) takes the form

$$u(p, 0) = p\Phi(p) + \Phi'(p) \quad \text{for all } p \in \mathbb{R}. \quad (3.8)$$

Now we will try to find the usual heat extension of $u(p, 0)$ (call it $\tilde{u}(p, t)$) which satisfies $\tilde{u}_{pp} = \tilde{u}_t$, and then we try to consider the formal candidate $u(p, t) := \tilde{u}(p, -t)$.

It is easier to find the heat extension of $\tilde{u}_p(p, 0)$ and then take the antiderivative in p . Indeed, notice that (3.8) implies $u_p(p, 0) = \Phi(p)$. the heat extension of $\Phi(p)$ is $\Phi\left(\frac{p}{\sqrt{1+2t}}\right)$. Indeed, the heat extension of the function $\mathbb{1}_{(-\infty, 0]}(p)$ at time $t = 1/2$ is $\Phi(p)$. Then by the semigroup property the heat extension of $\Phi(p)$ at time t will be the heat extension of

$\mathbb{1}_{(-\infty, 0]}(p)$ at time $1/2 + t$ which equals to $\Phi\left(\frac{p}{\sqrt{1+2t}}\right)$. Thus $\tilde{u}_p(p, t) = \Phi\left(\frac{p}{\sqrt{1+2t}}\right)$. Taking antiderivative in p and using (3.8) if necessary we obtain

$$\tilde{u}(p, t) = \sqrt{1+2t} \Phi' \left(\frac{p}{\sqrt{1+2t}} \right) + p \Phi \left(\frac{p}{\sqrt{1+2t}} \right).$$

This expression is well defined even for $t \in (-1/2, 0)$. Therefore if we set

$$u(p, t) = \tilde{u}(p, -t) = \sqrt{1-2t} \Phi' \left(\frac{p}{\sqrt{1-2t}} \right) + p \Phi \left(\frac{p}{\sqrt{1-2t}} \right) \quad \text{for } p \in \mathbb{R}, \quad t \in \left[0, \frac{1}{2}\right),$$

direct computations show that $u(p, t)$ satisfies (3.5), (3.8) and (3.7) because $\det(\text{Hess } u) = -\left(\frac{\Phi'(\frac{p}{\sqrt{1-2t}})}{1-2t}\right)^2 < 0$. We have $u_t = -\frac{\Phi'(\frac{p}{\sqrt{1-2t}})}{\sqrt{1-2t}} < 0$ and $u_p = \Phi\left(\frac{p}{\sqrt{1-2t}}\right)$. Therefore,

$$\begin{cases} x = \Phi\left(\frac{p}{\sqrt{1-q^2}}\right); \\ y = \frac{-q}{\sqrt{1-q^2}} \Phi'\left(\frac{p}{\sqrt{1-q^2}}\right); \end{cases} \quad \text{then} \quad \begin{cases} \Phi^{-1}(x) = \frac{p}{\sqrt{1-q^2}}; \\ y = \frac{-q}{\sqrt{1-q^2}} \Phi'(\Phi^{-1}(x)). \end{cases}$$

From the last equalities we obtain $M_y = q = -\frac{y}{\sqrt{I^2(x)+y^2}}$ and $M_x = p = \frac{I(x)\Phi^{-1}(x)}{\sqrt{I^2(x)+y^2}}$ where we remind that $I(x) = \Phi'(\Phi^{-1}(x))$. Then it is clear that

$$M(x, y) = -\sqrt{I^2(x) + y^2}.$$

4. ONE DIMENSIONAL CASE

Let $n = 1$, and set $\alpha = (\alpha_0, \dots, \alpha_m)$ where $\alpha_0 = 0, \alpha_1 = 1, \dots, \alpha_n = n$. Take $\mathbf{f} = \underbrace{(f, \dots, f)}_{m+1}$,

where $f \in C_0^\infty(\mathbb{R}; \Omega)$. Then $D^\alpha \mathbf{f} = (f, f', f'', \dots, f^{(m)})$. Given a log-concave probability measure $e^{-U(x)} dx$ such that $U''(x) \geq R > 0$, the associated semigroup P_t has the generator $L = d^2x - U'(x)dx$. Let $\mathbf{u} = (u_0, \tilde{\mathbf{u}})$ where $\tilde{\mathbf{u}} = (u_1, \dots, u_m) \in \mathbb{R}^m$ is arbitrary and $u_0 \in \Omega$. Let the function $B(u_0, \dots, u_m) \in C^2(\Omega \times \mathbb{R}^m)$. Let $B_j := \frac{\partial B}{\partial u_j}$ and $B_{ij} := \frac{\partial^2 B}{\partial u_i \partial u_j}$. Set

$$L_j(\mathbf{u}, y) = \sum_{k=j+1}^m \binom{k}{j} B_k(\mathbf{u}) U^{(k-j+1)}(y) \quad \text{for } j = 0, \dots, m-1$$

Remark 4. Notice that if $e^{-U(x)} dx = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ then $L_j(\mathbf{u}, y) = (j+1)B_{j+1}(\mathbf{u})$.

Further we assume that $B_{mm} \neq 0$. Theorem 2 implies the following corollary:

Corollary 2. The following conditions are equivalent:

(i) For all $\mathbf{u} \in \Omega \times \mathbb{R}^m$ we have

$$B_{mm} \leq 0, \quad \tilde{\mathbf{u}} \{B_{mj}(\mathbf{u}) B_{mi}(\mathbf{u}) - B_{mm}(\mathbf{u}) B_{ij}(\mathbf{u}) - \delta_{i-j} \frac{B_{mm}}{u_{j+1}} L_j(\mathbf{u}, y)\}_{i,j=0}^{m-1} \tilde{\mathbf{u}}^T \leq 0.$$

(ii) For all $f \in C_0^\infty(\mathbb{R}^m; \Omega)$ and $t \geq 0$ we have

$$P_t B(f, f', \dots, f^{(m)}) \leq B(P_t f, P_t f', \dots, P_t f^{(m)}).$$

Remark 5. If we send $t \rightarrow \infty$ then (ii) in the corollary implies an inequality

$$\int_{\mathbb{R}} B(f, f', \dots, f^{(m)}) d\mu(x) \leq B\left(\int_{\mathbb{R}} f d\mu, 0, \dots, 0\right) \quad \text{for all } f \in C_0^\infty(\mathbb{R}^m; \Omega).$$

Proof. It is enough to show that (i) in Corollary 2 is the same as (i) in Theorem 2. Notice that

$$[L, D^{\alpha_0}] = 0 \quad \text{and} \quad [L, D^{\alpha_k}] = \sum_{\ell=1}^k \binom{k}{\ell} U^{(\ell+1)}(x) d^{k+1-\ell} x \quad \text{for } 1 \leq k \leq m.$$

Thus

$$\nabla B[L, D^\alpha] \mathbf{f} = \sum_{k=1}^n \frac{\partial B}{\partial u_k} \left(\sum_{\ell=1}^k \binom{k}{\ell} U^{(\ell+1)}(x) (P_t f)^{k+1-\ell} \right)$$

and

$$\Gamma(D^\alpha f) = \begin{bmatrix} g' \cdot g' & g' \cdot g'' & \dots & g' \cdot g^{(m+1)} \\ g'' \cdot g' & g'' \cdot g'' & \dots & g'' \cdot g^{(m+1)} \\ \dots & \dots & \dots & \dots \\ g^{(m+1)} \cdot g' & g^{(m+1)} \cdot g'' & \dots & g^{(m+1)} \cdot g^{(m+1)} \end{bmatrix}$$

Therefore quantity (i) in Theorem 2 takes the following form

$$\sum_{k=1}^m B_k(\mathbf{u}) \left[\sum_{\ell=1}^k \binom{k}{\ell} U^{(\ell+1)}(y) u_{k+1-\ell} \right] + \sum_{i,j=0}^m B_{ij}(\mathbf{u}) u_{i+1} u_{j+1}$$

where u_1, \dots, u_{n+1}, y are arbitrary real numbers and u_0 takes values in Ω . Notice that the above expression can be rewritten as follows

$$B_{mm} u_{m+1}^2 + 2u_{m+1} \left(\sum_{j=0}^{m-1} B_{mj} u_{j+1} \right) + \sum_{i,j=0}^{m-1} B_{ij} u_{i+1} u_{j+1} + \sum_{k=1}^m B_k(\mathbf{u}) \left[\sum_{\ell=1}^k \binom{k}{\ell} U^{(\ell+1)}(y) u_{k+1-\ell} \right]$$

This expression is nonpositive if and only if condition (i) of Corollary 2 holds. \square

5. FURTHER APPLICATIONS

Houdré-Kagan [27] obtained an extension of the classical Poincaré inequality:

$$\sum_{k=1}^{2d} \frac{(-1)^{k+1}}{k!} \int_{\mathbb{R}^n} \|\nabla^k f\|^2 d\gamma \leq \int_{\mathbb{R}^n} f^2 d\gamma - \left(\int_{\mathbb{R}^n} f \right)^2 \leq \sum_{k=1}^{2d-1} \frac{(-1)^{k+1}}{k!} \int_{\mathbb{R}^n} \|\nabla^k f\|^2 d\gamma \quad (5.1)$$

for all compactly supported functions f on \mathbb{R}^n , and any $d \geq 1$. Here by symbol $\|\nabla^k f\|$ we denote

$$\|\nabla^k f\|^2 = \sum_{|\alpha|=k} (D^\alpha f)^2.$$

We refer the reader to [28] for further remarks on (5.1) in one dimensional case $n = 1$.

We will illustrate the use of Corollary 2 on (5.1) in case $n = 1$.

Proof of (5.1) in case $n = 1$.

Consider

$$B(u_0, u_1, \dots, u_m) = \sum_{k=0}^m \frac{(-1)^k}{k!} u_k^2,$$

and $d\mu = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$. If m is odd then $B_{mm} \leq 0$ and condition (i) of Corollary 2 holds. Indeed, in this case $L_j(\mathbf{u}, y) = B_{j+1}(\mathbf{u})(j+1) = u_{j+1}(-1)^{j+1}\frac{2}{j!}$, and

$$\begin{aligned} & \tilde{\mathbf{u}}\{B_{mj}(\mathbf{u})B_{mi}(\mathbf{u}) - B_{mm}(\mathbf{u})B_{ij}(\mathbf{u}) - \delta_{i-j}\frac{B_{mm}}{u_{j+1}}L_j(\mathbf{u}, y)\}_{i,j=0}^{m-1}\tilde{\mathbf{u}}^T = \\ & - B_{mm}\tilde{\mathbf{u}}\left\{B_{jj} + B_{j+1}\frac{j+1}{u_{j+1}}\right\}_{i,j=0}^{m-1}\tilde{\mathbf{u}}^T = 0. \end{aligned}$$

Thus by (ii) of Corollary 2 we obtain that for all $f \in C_0^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} \sum_{k=0}^m \frac{(-1)^k}{k!} [f^{(k)}(x)]^2 d\mu \leq \left(\int_{\mathbb{R}} f(x) d\mu \right)^2$$

for odd m , and similarly we obtain the opposite inequality for even m .

Proof of (1.22) (Banaszczyk conjecture). We will show that if f is even and $d\mu = e^{-U(x)}dx$ is an even log-concave measure such that $\text{Hess } U \geq \text{Id}$ then

$$\left(\int_{\mathbb{R}^n} f^2 d\mu \right)^2 - \left(\int_{\mathbb{R}^n} f d\mu \right)^2 \leq \frac{1}{2} \int_{\mathbb{R}^n} \|\nabla f\|^2 d\mu. \quad (5.2)$$

Indeed, take $M(x, y)$ as in (1.23) i.e.,

$$M(x, y) = x^2 - \frac{y^2}{2} \quad \text{for } x \in \mathbb{R}, \quad y \geq 0.$$

Unfortunately $M(x, y)$ does not satisfy (1.1) (because $M_{xx} + M_y/y = 1 > 0$) therefore we cannot directly apply Theorem 1.

Let P_t be the associated semigroup to $d\mu$ and let L be its generator. Consider the function $V(x, t) = P_t M(f, \|\nabla f\|) - M(P_t f, \|\nabla P_t f\|)$ as in the proof of Theorem 2. Then

$$\begin{aligned} (\partial_t - L)V(x, t) &= -\nabla P_t f (\text{Hess } U) (\nabla P_t f)^T + 2\|\nabla P_t f\|^2 - \|\nabla^2 P_t f\|^2 \leq \\ & \|\nabla P_t f(x)\|^2 - \|\nabla^2 P_t f\|^2. \end{aligned}$$

Clearly it is not true that the above expression is pointwise i.e., for all $x \in \mathbb{R}^n$, non positive (consider $t = 0$). Therefore we cannot directly apply maximum principle as in the proof of Theorem 2 in order to get pointwise bound $V(x, t) \leq 0$. Actually we do not need pointwise estimate $V(x, t) \leq 0$ in order to get (5.2), for example $\int_{\mathbb{R}^n} V(x, t) d\mu \leq 0$ will be enough. Notice that

$$\int_{\mathbb{R}^n} V(x, T) d\mu = \int_0^T \int_{\mathbb{R}^n} (\partial_t - L)V(x, t) d\mu dt \leq \int_0^T \int_{\mathbb{R}^n} \|\nabla P_t f(x)\|^2 - \|\nabla^2 P_t f\|^2 d\mu ds \leq 0$$

for all $T \geq 0$. The last inequality follows from the application of Poincaré inequality (1.10) to the functions $\partial_{x_j} P_t f(x)$ for all $j = 1, \dots, n$, and the fact that $\int_{\mathbb{R}^n} \partial_{x_j} P_t f(x) d\mu = 0$ because $P_t f(x)$ is even function. Thus we obtain that

$$\int_{\mathbb{R}^n} M(f, \|\nabla f\|) d\mu \leq \int_{\mathbb{R}^n} M(P_T f, \|\nabla P_T f\|) d\mu \quad \text{for all } T \geq 0.$$

By sending $T \rightarrow \infty$ we arrive at (5.2) because $\lim_{T \rightarrow \infty} \|\nabla P_T f\| = 0$. □

In the end we should mention that even though the current paper is self-contained it should be considered as a continuation of the ideas developed in our recent papers [5, 6] where similar to (1.1) PDEs happen to rule some functional inequalities.

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DEPARTMENT OF MATHEMATICS, KENT STATE UNIVERSITY, KENT, OH 44240, USA
E-mail address: `ivanishvili.paata@gmail.com`

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824, USA
E-mail address: `volberg@math.msu.edu`